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A Review of Models for Dependency of Risks: Extension and Applications to Supply Chains

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ABSTRACT

Today's highly integrated supply chains are exposed to various types of risks that disrupt the normal flow of goods or services within a supply chain network. Since most of these individual risks are interconnected, a mitigation strategy to tackle one risk may result in the exacerbation of another. Given that the occurrence of one risk may cause a chain reaction, an important question arises: how to model risk dependencies in a supply chain and what factors are relevant in measuring supply chain dependencies? In the financial insurance literature, risk dependencies have been modeled using two approaches: (i) random variables, and (ii) copulas. This paper first reviews these studies to understand the dependency factors and their sources. Then, these models are extended for predicting and mitigating supply chain risks under dependencies. Finally, those models are applied to different supply chain network configurations.

Keywords: *risk dependency, risk modelling, risk management, supply chain*

1. INTRODUCTION

Today's supply chains are challenged by globalization, complex networks, rapid technological changes, and other factors, leading to increased uncertainties and risks (Wang *et al*., 2015). Planning and assessment of strategies to mitigate these risks are essential stages in managing supply chain risks (Sumarliah *et al*., 2021; Waqas *et al*., 2023). Several review papers discussing methodologies for risk assessment in supply chains can be found in the works of Hudnurkar *et al*. (2017), Choudhary *et al*. (2023), and Wang *et al*. (2020). As supply chain networks are becoming more connected and interdependent, a failure at any point of the network can result in major disruptions to the flow of products or services in the network, which implies risks in supply chains are interconnected. For example, forest fires in one region may block highways and thus prevent a supplier from shipping material to a manufacturer in another region. This can, in turn, delay shipping products to a retailer, thus creating a risk-cascading effect. There are different types of risks reflecting different components and processes in the supply chain, as listed in **Table 1**. Furthermore, the increase in complexity and offshoring of products and services has increased the risk diversity within the supply networks (Harland *et al*., 2003; Ohmori *et al*., 2023). We would expect these risks to be more dependent and their occurrence may cause a chain reaction. This then raises the important question of how to model risk dependencies in a supply chain and what factors are relevant in measuring supply chain dependencies. Understanding these dependency factors and their sources would help us in building models for predicting and mitigating supply chain risks under such dependencies. In addition, the fact that supply chain operational models had traditionally emphasized economic efficiency, leading to supply chains operating on minimum inventories and facilities, is likely to delay the recovery from disruption (Jones, 2013). The use of just–in–time systems had meant that supply chains reduced their suppliers pool and in turn competing supply chains often share the same supplier (Jones, 2013). Thus, a disruption in one supply point can lead to simultaneous disruptions in several supply chains. After the 2011 earthquake in Japan, the major auto manufacturers have started questioning the way they operate their supply chains and how to incorporate risk effects in their design and operation. For example, Aston Martin and Jaguar Land Rover have partnered with Toyota to share information about their supply chain networks in the hope that they can anticipate risky events before they occur (Jones, 2013). This trend for collaboration was mostly driven by the realization that their supply networks are interconnected. As Guillaume Jacques, a purchasing manager at Toyota Motor Europe, said (Jones, 2013):

Our supply chains are so interdependent that there is no point in Toyota trying to secure its supply chain on its own. Any manufacturer stopping production on a big scale would impact others within a very short time.

Thus, most of the individual risks are interconnected, so a mitigation strategy to tackle one risk may result in exacerbation of another. Furthermore, the existing models in the literature, mainly from the finance and insurance fields, mostly focus on single risk events and when they consider multiple events, they often assume that they are independent.

We review the finance and insurance models on risk management with a focus on dependency modelling and show, when possible, how they can be applied in a supply chain context. We also outline the limitations of these models and present an extension that is relevant for supply chains' risk analysis.

The outline of this paper is as follows: First, we describe the required mathematical concepts in Section 2. Prevalent terms in the insurance context are also defined in this section. Types of supply chain risks as well as events and conditions that drive them are summarized in Section 3. We discuss the source of dependencies in supply chains at the end of Section 3. Risk dependencies models are then described in Sections 4–6 using two approaches: random variables and copulas. In each category (random variables and copulas) individual risks are either in a single class or divided in different classes. Furthermore, we outline the limitations of these models and present the applications of dependency models, when possible, in a supply chain context. In Section 7, we relax one of the critical assumptions of the existing models, that of individual risk independence, and present an extension, considering dependency among individual risk factors. Finally, in Section 8 we provide a summary of the discussed models and discuss some future research directions.

2. RISK DEPENDENCY MODELLING TERMS AND APPROACHES

The models we review borrow heavily from the insurance mathematics literature, the terminology of which may not be familiar to the operations research and management science researchers. In this section, we define several terms and concepts that we use in this paper. The interested reader can find more details in Bauerle and Muller (1998), Denuit *et al*. (2006) and Shaked and Shanthikumar (1997).

An insurance contract (policy) is a legally binding agreement between the insurer and insured that contains terms and conditions of the insurance coverage. An insurance premium is the financial cost of an insurance policy, paid by a policy holder either as a lump sum or in several instalments during the period covered by the policy. A collection of different risks caused by policy holders (insured) is referred to as a risk portfolio in the insurance literature. For example, risks like illnesses or accidents which threat an individual's life in life insurances. A quasi-homogeneous portfolio is a portfolio that is homogeneous with respect to claim amounts distributions, but heterogeneous with respect to claim occurrences, where the policy holders would have different occurrence probabilities. A risk portfolio can ve divided into several classes where the insurers are classified according to the risk they present for the insurer. Models that address these types of risk portfolios are referred to as multi-class risk models.

Supermodular theory is used for comparing the riskiness of different risk portfolios. A function $f: R^n \rightarrow R$ is said to be supermodular, if

$$\begin{aligned} f(x_1, \cdots, x_i + \varepsilon, \cdots, x_j + \delta, \cdots, x_n) \\ - f(x_1, \cdots, x_i + \varepsilon, \cdots, x_j, \cdots, x_n) \\ \geq f(x_1, \cdots, x_i, \cdots, x_j + \delta, \cdots, x_n) \\ - f(x_1, \cdots, x_i, \cdots, x_j, \cdots, x_n) \end{aligned}$$

holds for all $x \in R^n$, $1 \leq i < j \leq n$ and all $\varepsilon, \delta > 0$. The function f is symmetric, if $f(x) = f(\pi x)$ for all permutations πx of x. A random vector $X = (X_1, \ldots, X_n)$ is said to be smaller than the random vector $Y = (Y_1, \ldots, Y_n)$ in the (symmetric) supermodular ordering, written $X \leq_{(sym)sm} Y$, if $E[f(X)] \leq E[f(Y)]$ for all (symmetric) supermodular functions f assuming expectations exist. In addition, the concept of a stop-loss order is used to compare risk portfolios. For arbitrary univariate random variables Y we denote the stop-loss transform $\pi_Y(t) = E(Y-t)^+ = \int_t^\infty \bar{F}_Y(x)dx, t \in R$. We say X precedes Y in stop-loss order, written $X \leq_{sl} Y$, if $\pi_X(t) \leq \pi_Y(t)$ for all $t \in R$.

Stochastic risk dependencies have been modelled by random variables and distribution funtions (copula). Latent random variables are random variables that are not directly observed. Their properties must thus be inferred indirectly using a statistical model (latent variable model) connecting the latent (unobserved) variables to observed variables. If there are two possible values α_1 and α_2 in an experiment, then we can use a two-point probability distribution to describe it as follow: $\Pr(X = \alpha_1) = p$ and $\Pr(X = \alpha_2) = 1 - p = q$ $p, q \geq 0$, $p + q = 1$. A mixture distribution has a density function f that is a convex combination of probability density functions g_i, i.e., $f(x) = \sum_{i=1}^n p_i g_i(x)$, $\sum_{i=1}^n p_i = 1, n > 1$. In cases where each of the underlying distribution functions is continues, the outcome mixture distribution function would be continuous as well. When g_i's are from a parametric family with unknown parameters θ_i's, the parametric mixture distribution is defined as $\sum_{i=1}^n p_i g(x|\theta_i)$. A prior probability distribution, or the prior, of an uncertain quantity p, a parameter or a latent variable, is the probability distribution that would express one's uncertainty about p before known data are taken to account. De Finetti's Theorem can be used to establish the existence of a prior for a random parameter Θ. According to De Finetti's Theorem, to every infinite sequence of exchangeable random variables $\{X_n\}$ having values in $\{0, 1\}$, there corresponds a probability F over $[0, 1]$, such that, $p_{k,n} = P(X_1 = 1, \ldots, X_k = 1, \ldots, X_{k+1} = 0, X_n = 0) = \int_0^1 \theta^k (1-\theta)^{(n-k)} F(d\theta)$. In particular, the expression for $p_{k,n}$ holds when Θ has prior F and given $\Theta = \theta, X_1, X_2, \ldots$ are independent Bernoulli, with parameter θ. A copula function is a joint distribution function with marginal distribution functions as parameters. Therefore, properties of copulas are similar to those of joint distributions. The Frechet-Hoeffding Theorem states that for any copula $C: [0,1]^d \rightarrow [0,1]$ and any $(u_1, \ldots, u_d) \in [0,1]^d$, C is bounded from below and above as follows: $W(u_1, \dots, u_d) \leq C(u_1, \dots, u_d) \leq M(u_1, \dots, u_d)$.

The function W is referred to as the lower Frechet-Hoeffding bound and is defined as $W(u_1, \dots, u_d) = \max\{1 - d + \sum_{i=1}^{d} u_i, 0\}$. The function M is called the upper Frechet-Hoeffding bound and is defined as $M(u_1, \dots, u_d) = \min\{u_1, \dots, u_d\}$.

3. RISK IN SUPPLY CHAINS

3.1 Types of Risks

As the vulnerability of a supply chain to disruption increases, it is important to identify and manage various types of risks to avoid any supply chain breakdown. Supply chain risks can penetrate every stage, not just the final stage of product/service delivery to customers. On the other hand, dependence on different business partners in each stage of production can worsen the impact of risk on the supply chain. So, a way of reducing damage driven by risks is to determine factors which cause dependencies in a supply chain.

Supply chain risks can be divided into different types based on their realization impact on a business and its environment. A summary of the most important risks and examples can be found in **Table 1.** Some of these risks are mentioned by Chopra and Sodhi (2004) and Harland *et al.* (2003).

Supply chain risks can also be classified based on whether they impact strategic or operational supply chain decision making. In this context we define strategic supply chain risks as those the impact of which will be long-term causing the supply chain decision makers to change their strategics. For example, the earthquake in Japan has led major auto manufacturers, such as Toyota, to rethink their risk management strategies. On the other hand, an operational supply chain risk has a short-term impact on the supply chain operations. An example is a temporary machine breakdown or a limited disruption that may occur due to an employee leaving a company. In addition, supply chain risks can be classified based on their source, upstream, internal, or downstream. In **Figure 1** we propose a strategy-source matrix for classifying the different supply chain risks.

Table 1 Supply chain risks

Type of Risks	Examples
Disruption	Natural disasters, labor strike, fires, and terrorism are examples of disruptions which can interrupt the flow of the material in a supplier or a manufacturer. Based on the effect, disruptions can be operational or strategic.
Quality	Poor quality in a supply source or in a manufacturer's product or even any failure arising from customers can harm the quality of supply chain's products and/or services. Based on the duration of the impact, a quality risk can be operational or strategic.
Forecast	Long lead times, seasonality, product variety, short life cycles, small customer base, information falsification due to promotion, intensive, lack of supply-chain visibility and exaggeration of demand in times of product shortage, can cause an inaccurate forecast.
Legal	Any action arising from suppliers, customers, shareholders or employees which expose a firm to judicial process.
Reputation	This strategic risk is one of the most critical ones. Loss of confidence can destroy the whole value of a business financial.
Receivables	When a company is not able to collect the receivables, its performance will be affected.
IT	Any breakdown of information basis can destroy the highly networked environment of a company.
Capacity	Increasing or decreasing the capacity can be a strategic decision for companies. Having too much of excess capacity can negatively affect a company's financial performance.
Market	Changes in demand of customers, sticking to a single marketing strategy, exchange rate risk and interest rate risk are examples of market risks.
Financial	Failure of debtors to meet their financial obligation and changes in financial markets can cause a financial loss for a company.
Competitive risk	An example is when a firm is not able to differ its products/services form the other competitors.
Human Resources	Businesses face a strategic risk when a key executive leaves the company.

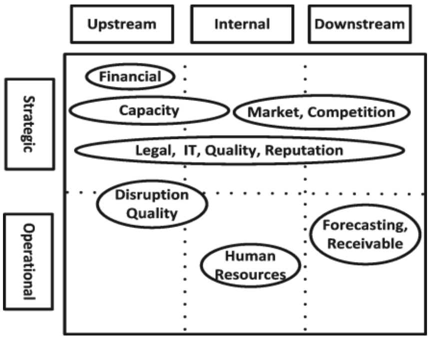

Figure 1 Strategy-source matrix for classifying supply chain risks

The matrix in **Figure 1** can be used to help in prioritizing risk planning and mitigation strategies. For example, we can see that Legal, IT, Quality and Reputation may involve all links in a supply chain. We would then expect these risks to be more dependent and their occurrence may cause a chain reaction. This then raises the important question of how to model risk dependencies in a supply chain and what factors are relevant in measuring supply chain dependencies. In the following section some factors which have effect on the amount of dependence between supply chain links will be examined.

3.2 Types of Supply Chain Dependencies

Dependence between links in a supply chain can be a function of different factors such as the type of relationship (transactional, preferential or strategic), the criticality of the item in the final product, spend value, number of suppliers, and location of suppliers. In **Table 2** we list different factors that may cause supply chain dependencies together with their sources. Understanding these dependency factors and their sources would help us in building models for predicting and mitigating supply chain risks under such dependencies.

In the next section we focus on mathematical models for representing risk dependencies and indicate how they apply to supply chains.

Table 2 Sources of dependence in a supply chain

Source of Dependences	Examples
Sourcing	■ Number of suppliers ■ Location of supplier ■ Local or global sourcing ■ Supplier dependence ■ Supplier commitment to buyer ■ Supplier power ■ Financial strength ■ Type of contract
Distribution	■ Distribution network ■ Transportation network ■ Transportation modes ■ Rely on technologically advanced (key) suppliers
Customer	■ Volume ■ Loyalty ■ Customer-driven supply-chain
Information	■ Shared information ■ Push vs. pull ■ Information Technology service capabilities

4. RANDOM VARIABLE-BASED DEPENDENT SINGLE-GROUP RISK MODELS

In this section we look at two models that employ random variables to represent risk dependencies within a single group. The first model used a compound Poisson random variable and the second used a two-point random variable.

4.1 Compound Poisson Approximation Model (Goovaerts and Dhaene (1996))

Consider a portfolio consisting of n dependent risks. With risk k we associate a claim amount X_k represented as $X_k = J_k B_k$, where J_k is a Bernoulli random variable which is equal to 1 if risk k causes at least one claim during the reference period with probability q_k, i.e., $\Pr(J_k = 1) = q_k = 1 - \Pr(J_k = 0)$, and B_k is the total claim amount produced by risk k. Dependence in this model is represented by the dependent indicators J_k. However, the conditional claim amounts B_k such that $J_k = 1$ (denoted by $B_k \mid J_k = 1$) are mutually independent.

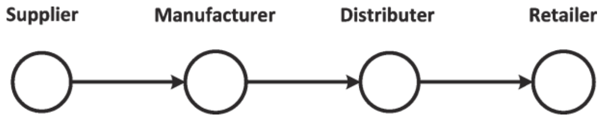

Figure 2 A serial supply chain

The total claim of the portfolio during a fixed period of time is $S = X_1 + X_2 + \ldots + X_n$ and its cumulative distribution function is $F(s) = \Pr(S \le s)$. If we assume that all conditional claim amounts $B_k \mid J_k = 1$ have the same cumulative distribution F, the portfolio is quasi-homogeneous and thus F can be approximated with a compound Poisson with cumulative distribution function F^{cp} as follows:

$$F^{cp}(s) = \sum_{n=0}^{s} \Pr(K = n)\, F^{n}(s) \qquad (s = 0, 1, \ldots)$$

Where K is a Poisson distribution random variable with parameter λ given by $\lambda = \sum_{k=1}^{n} q_k$ and $F(s)$ is the distribution given by:

$$F(s) = \frac{1}{\lambda} \sum_{k=1}^{n} q_k \Pr(B_k \le s \mid J_k = 1).$$

It is worth noting that the above approximation does not hold when all conditional claim amounts $B_k \mid J_k = 1$'s are dependent.

Application of Model 4.1 in Supply Chains:

One possible application of this model is in serial supply chains (see **Figure 2**). Here we assume that $J_k = 1$ if risk k affects at least one of the suppliers, manufacturer, distributer or retailers. The random variable B_k represents the total loss produced by risk k. Since in a serial supply chain each partner has at most one predecessor and one successor, the risk dependency is limited to such interactions. These supply chains exist where a business is dealing with simple products that do not require much value adding, except for inventory and distribution. An example can be found in produce supply chains where a farmer produces and packages fruits or vegetables and sells them directly to consumers (pick your own) or through a distributor and/or retailer.

4.2 Two-Point Distribution Model (Dhaene and Goovaerts (1997))

Assume that n risks $X_1, X_2, \ldots, X_n$ form a portfolio. In this model each risk X_k $(k = 1, \ldots, n)$follows a two-point distribution in 0 and $\alpha_k \geq 0$ denoted by $\Pr(X_k = 0) = p_k$ and $\Pr(X_k = \alpha_k) = 1 - p_k = q_k$.

If random variables $X_1, \ldots, X_n$ are assumed mutually independent, cumulative distribution function of the total claims $S = X_1 + X_2 + \ldots + X_n$ is uniquely determined by the distributions of the X_i's (using convolution of n probability mass functions). But, in this model we will not have the assumption of independence.

The expected aggregate claim will not be affected by the type of dependence between the individual risks because for each $S \in \Re$ (set $\Re$ is consisting of all random variables S that cab be written as $S = X_1 + X_2 + \ldots + X_n$) the mean value is calculated by:

$$E(S) = \sum_{k=1}^{n} q_k \alpha_k.$$

We will suppose that the individual risks are arranged in an increasing order $p_1 \leq p_2 \leq \ldots \leq p_n$, i.e., the risk with a lower subscript has a lower probability. The dependence between the individual risks is given by the following relation:

$$\Pr(X_{k+1} = 0 | X_k = 0) = 1, k = 1, 2, \ldots, n - 1. \quad (1)$$

It follows from Eq. (1) that:

$$\Pr(X_{k+1} = 0) = \Pr(X_{k+1} = 0 | X_k = 0) \Pr(X_k = 0) + P(X_{k+1} = 0 | X_k = \alpha_k) \Pr(X_k = \alpha_k)$$
$$p_{k+1} = 1 \times p_k + \Pr(X_{k+1} = 0 | X_k = \alpha_k) \times (1 - p_k)$$

Therefore, we have $\Pr(X_{k+1} = 0 | X_k = \alpha_k) = \frac{p_{k+1} - p_k}{1 - p_k}$, $\Pr(X_{k+1} = \alpha_{k+1} | X_k = 0) = 0$, and $\Pr(X_{k+1} = \alpha_{k+1} | X_k = \alpha_k) = \frac{1 - p_{k+1}}{1 - p_k}$.

This model has been developed for the analysis of life insurance claims. From (1) it follows that if person k stays alive then person $k + 1$ stays alive, but if person $k + 1$ stays alive then person $k + 2$ stays alive, and so on. So we can conclude:

$$\Pr(X_{k+j} = 0 | X_k = 0) = 1, k = 1, 2, \ldots, n - 1; \quad j = 1, \ldots, n - k.$$

This means that "if a person survives the exposure period, then all persons with greater survival probabilities will also survive" (Dhaene and Goovaerts, 1997).

Theorem 4.2.1. Considering all former relations, the possible outcome for S will be: $0, \alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \ldots, \alpha_1 + \ldots + \alpha_n$. Then the distribution function of S will be as follow:

$$F_s(s) = \begin{cases} p_1, & 0 \leq s \leq \alpha_1 \\ p_{k+1}, & \alpha_1 + \cdots + \alpha_k \leq s \leq \alpha_k + \cdots + \alpha_{k+1}, \\ 1, & \alpha_1 + \cdots + \alpha_k \leq s \leq \alpha_k + \cdots + \alpha_{k+1} \\ & k = 1, 2, \ldots, n - 1 \end{cases}$$

Proof: See the appendix.

Model 4.2 does not seem very practical compared to Model 4.1, since dependency between risks $X_k (k = 1, \ldots, n)$, as expressed by (1), leads to the riskiest portfolio in the sense that it has the largest stop-loss premium. (Dhaene and Goovaerts, 1997).

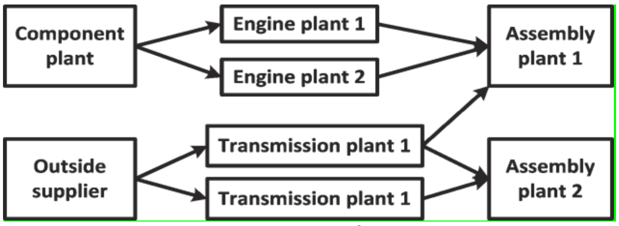

Figure 3 Schematic representation of model 4.2 in supply-chain (Deleris et al., 2004)

Application of Model 4.2 in Supply Chains:

An assembly network would be a possible application of Model 4.2. In **Figure 3** the outside supplier is a single source to the two downstream transmission plants. If a disruption happens in the outside supplier, even if partial, all downstream plants, including the assembly plants may be influenced to different degrees. The latter reflects the condition $\Pr(X_{k+1} + 1 = 0 | X_k = 0) = 1$ in model 4.2.

5. RANDOM VARIABLE-BASED DEPENDENT MULTI-GROUP RISK MODELS

In this section we continue to discuss random variable-based models but for multi-group situations. We describe six models that incorporate risks due to different group factors.

5.1 Risk as a Function of Three Risk Factors

In the first model of Bauerle and Muller (1998), there is a strong dependence between members of one group of a portfolio, but the dependence between members of different groups is weaker.

Consider portfolio $X = (X_1, \ldots, X_n)$, consisting of n risks $X_1, \ldots, X_n$. Moreover, we assume that there exists an increasing function $g : R^3 \rightarrow R$ such that the k-th risk is given by $X_k = g(Z_k, G_r, V)$ where k is in group r, V is an overall risk factor, G_r is a group-specific risk factor which impacts only the risks in group r and Z_k is an individual risk factor which indicates the individual share of risk X_k, $1 \leq k \leq n$. In general, comparing two risky portfolios with different sizes and number of groups is difficult. However, using **Theorem 5.1.2** (that will be described later in this section) makes the comparison possible in some cases. In order to state the theorem, let L and L′ be two n-dimensional vectors with $L = (L_1, \ldots, L_r, 0, \ldots, 0)$, $L = (L'_1, \ldots, L'_l, 0, \ldots, 0), 1 \leq r, l \leq n, L_k, L'_k \in N$ for all k and $\sum_{k=1}^{n} L_k = \sum_{k=1}^{n} L'_k = n$. Then, we consider two n-dimensional risky portfolios X and Y given by

$$X_1 = g(Z_1,\ G_1, V) \qquad Y_1 = g(U_1, G_1, V)$$
$$\vdots \qquad\qquad \vdots$$
$$X_{L_1} = g(Z_{L_1},\ G_1, V) \qquad Y_{L'_1} = g(U_{L'_1}, G_1, V)$$
$$X_{L_1+1} = g(Z_{L_1+1},\ G_2, V) \qquad Y_{L'_1+1} = g(U_{L'_1+1}, G_2, V)$$
$$\vdots \qquad\qquad \vdots$$
$$X_{L_1+L_2} = g(Z_{L_1+L_2},\ G_2, V) \qquad Y_{L'_1+L'_2} = g(U_{L'_1+L'_2}, G_2, V)$$
$$\vdots \qquad\qquad \vdots$$
$$X_n = g(Z_n,\ G_r, V) \qquad Y_n = g(U_n, G_l, V)$$

Where the individual risk factors $Z_1, \dots, Z_n, U_1, \dots, U_n$ are i.i.d. random variables, the group specific risk factors $G_1, \dots, G_{max(r,l)}$ are i.i.d. random variables and the environmental risk factor V is a random variable independent of Z_k, U_k and $G_{max(r,l)}$. Let $S = \sum_{k=1}^{n} X_k$ and $S' = \sum_{k=1}^{n} Y_k$.

The theorem uses majorization (Marshall and Olkin, 1979) for comparing the structure of vectors L and L′. The definition of majorization is as follows

Definition 5.1.1. let $X, Y \in N_0^n$ and denote by $X_{[1]} \geq \dots \geq X_{[n]}$ the decreasing rearrangement of X, and similarly for Y. We say that Y majorizes X ($X \prec Y$) if and only if

$$\sum_{k=1}^{r} X_{[k]} \leq \sum_{k=1}^{r} Y_{[k]},\ r = 1, \dots, n-1 \text{ and } \sum_{k=1}^{n} X_{[k]} = \sum_{k=1}^{n} Y_{[k]}.$$

Intuitively speaking $X \prec Y$ means that in Y the groups are larger and/or more unequal. The following theorem of Bauerle and Muller (1998) states the main result for this model:

Theorem 5.1.2. [Bauerle and Muller (1998)] If $L \prec L'$, we get under the assumptions of Model 5.1:

a) $X \leq_{symsm} Y$
b) $S \leq_{sl} S'$

Where $\leq_{symsm}$ denotes symmetric supermodular ordering, and $\leq_{sl}$ denotes stop-loss ordering, as defined at the beginning of this chapter.

In this model we assume that all individual risk factors $Z_1, \dots, Z_n, U_1, \dots, U_n$ and all group risk factors are i.i.d. In practice it is possible to have dependent individual risk factors or dependent group risk factors.

Application of Model 5.1 in Supply Chains

This model applies to supply chain networks that have an unequal number of partners at each echelon and where one has a more extensive network than the other. In **Figure 4** network 2 has larger and more unequal groups. So, based on the result of Model 5.1 network 2 is riskier than network 1. The situation in **Figure 4** can result when looking at an alternative of a local manufacturing supply chain that is more streamlined (e.g., Tesla electric cars) and comparing it to a global manufacturing supply chain that has a wide distribution network (e.g., Toyota).

5.2 Risk Dependency through Global System Shocks (Genest at al. (2003))

Consider a portfolio of $m \geq 1$ classes including $n_1, \dots, n_m$ contracts, and X_{jk} is the risk related to the kth contract in the jth class. Therefore, the aggregate claim amount is given by:

$$S = \sum_{j=1}^{m} \sum_{k=1}^{n_j} X_{jk}.$$

A global shock, represented by an indicator random variable J_0, can impact whole parts of the portfolio. Risk in class j occurs with a probability that whether the overall (such as a disaster or

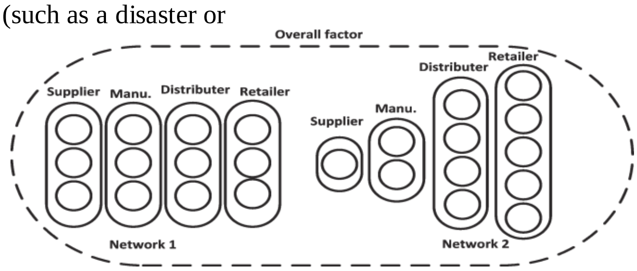

Figure 4 Schematic representation of an application of model 5.1 in supply chain. Each circle represents a company/facility

catastrophe) has occurred ($J_0 = 1$) or not ($J_0 = 0$). Let $J_j^{(\alpha)}$ imply the presence or absence of a global shock in class j, given $J_0 = \alpha$, ($\alpha = 1$ or 0). The risk related to the kth contract in the jth class for a fixed value of α and $\beta = J^{(\alpha)} \in [0, 1]$ is given by:

$$X_{jk}^{(\alpha\beta)} = J_{jk}^{(\alpha\beta)} B_{jk}^{(\alpha\beta)}$$

Where occurrence Bernoulli random variables J_{jk} is one when at least one claim is filed, and the amount of total claim associated to the kth contract in the *j*th class is indicated by strictly positive random variable B_{jk}.

Let $\bar{I} = 1 - I$ for any probability or indicator function I. The risk X_{jk} may then be indicated in the form:

$$X_{jk} = J_0\left(J_j^{(1)} X_{jk}^{(11)} + \bar{J}_j^{(1)} X_{jk}^{(10)}\right) + \bar{J}_0\left(J_j^{(0)} X_{jk}^{(01)} + \bar{J}_j^{(0)} X_{jk}^{(00)}\right). \qquad (2)$$

It sounds reasonable to assume that:

(a) J_0, the $J_j^{(\alpha)}$ ‘s and the $J_{jk}^{(\alpha\beta)}$ ‘s are mutually independent.
(b) $B_{jk}^{(\alpha\beta)}$ ‘s are independent of each other and of all indicator’s random variables.

In **Theorem 5.2.1.** we provide the distribution function $F_{X_{jk}}$ of X_{jk} from those of $X_{jk}^{(\alpha\beta)}$ ‘s and $B_{jk}^{(\alpha\beta)}$ ‘s.

Theorem 5.2.1. [Genest et al. (2003)] Define $E(J_0) = r$, $E\left(J_j^{(\alpha)}\right) = r_j^{(\alpha)}$ and $E\left(J_{jk}^{(\alpha\beta)}\right) = r_{jk}^{(\alpha\beta)}$ and similarly $\bar{r}, \bar{r}_j^{(\alpha)}$and, $\bar{r}_{jk}^{(\alpha\beta)}$. The distribution of the strictly positive random variable B_{jk} is a mixture distribution:

$$F_{B_{jk}} = \frac{r r_j^{(1)} r_{jk}^{(11)}}{q_{jk}} F_{B_{jk}^{(11)}} + \frac{r \bar{r}_j^{(1)} r_{jk}^{(10)}}{q_{jk}} F_{B_{jk}^{(10)}} + \frac{\bar{r} r_j^{(0)} r_{jk}^{(01)}}{q_{jk}} F_{B_{jk}^{(01)}} + \frac{\bar{r} \bar{r}_j^{(0)} r_{jk}^{(00)}}{q_{jk}} F_{B_{jk}^{(00)}}$$

Proof: See the appendix.

As X_{jk}'s are dependent on this model, it is hard to calculate directly the distribution F_S of the total claim amount S of the portfolio. But there is a handy way to avoid this problem. We can consider a mixture structure for S under conditions (a), (b) and (2). F_S can then be formulated in the following form, where Θ is a latent random vector with distribution M:

$$F_S(x) = \int F_{S^{(\theta)}}(x) dM(\theta),$$

Where $S^{(\theta)}$ is distributed as S given $\Theta = \theta$. To be clear, let $\Theta = (\Theta_0, \ldots, \Theta_m)$ have values in $\{0,1\}^{m+1}$ such that:

$$P(\Theta_0 = \alpha) = P(J_0 = \alpha) = r^{\alpha} \bar{r}^{(1-\alpha)}$$

and

$$P(\Theta_1 = \beta_1, \ldots, \Theta_m = \beta_m | \Theta_0 = \alpha)$$

$$= P\left(J_1^{(\alpha)} = \beta_1, \ldots, J_m^{(\alpha)} = \beta_m\right) = \prod_{j=1}^{m} \left(r_j^{(\alpha)}\right)^{\beta_j} \left(\bar{r}_j^{(\alpha)}\right)^{1-\beta_j}$$

for all $\alpha, \beta_1, \ldots, \beta_m \in \{0,1\}$. For $\Theta = \theta$, write $X_{jk}^{(\theta)} = X_{jk}^{(\alpha\beta_j)}$ and let:

$$S_j^{(\theta)} = \sum_{k=1}^{n_j} X_{jk}^{(\theta)}, \quad S^{(\theta)} = \sum_{j=1}^{m} S_j^{(\theta)} \qquad (4)$$

So, based on Eq. (2) we will have:

$$F_s(x) = P\left(\sum_{j=1}^{m} \sum_{k=1}^{n_j} X_{jk} \le x\right)$$

$$= \sum_{\alpha=0}^{1} \sum_{\beta_1=0}^{1} \cdots \sum_{\beta_m=0}^{1} P\left(\sum_{j=1}^{m} \sum_{k=1}^{n_j} X_{jk}^{(\alpha\beta_j)} \le x\right) P\{\Theta = (\alpha, \beta_1, \ldots, \beta_m)\}$$

$$= \sum_{\theta \in \{0,1\}^{m+1}} P\left(\sum_{j=1}^{m} S_j^{(\theta)} \le x\right) P(\Theta = \theta)$$

$$= \sum_{\theta \in \{0,1\}^{m+1}} F_{S^{(\theta)}}(x) P(\Theta = \theta)$$

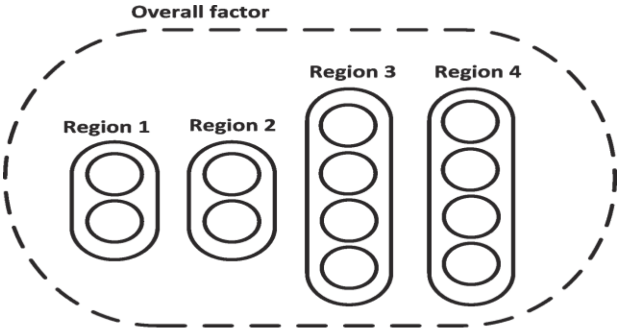

Figure 5 Schematic representation of model 5.2 in supply chains. Each circle represents a company/facility

which is the same as Eq. (3) for Θ discrete.

As $S_j^{(\theta)}$'s are mutually independent and each of them is a finite sum of mutually independent random variables, we can take advantage of the former representation for finding the distribution function F_S for a given θ.

In this model using a mixture structure for S is a useful way to express F_s when the X_{jk} is dependent. But we still do not know how to deal with cases in which the class and group risk factors and the B_{jk}'s are dependent.

Application of Model 5.2 in Supply Chains:

Figure 5 can be an example of Toyota's supply chain which operates its business worldwide. As explained in this model, companies (individual risk factors) that are in different regions (group risk factors) do not have any contribution with each other as they are assumed to be independent. But a consequence of Toyota's excessive expansion was that it became increasingly dependent on suppliers outside Japan, or Toyota branches in other countries became dependent on suppliers in Japan. For example, Toyota gets only 15 percent of its parts including electronic and rubber components from Japan for cars and trucks built in North America. But still, it has to have all of them to build a vehicle. It means that a shutdown of a part supplier in Japan can result in a shutdown of an assembly plant in North America. If the model 5.2 is applied to a supply chain network, then we can figure out the distribution function of the total loss caused by risks on companies involved in the network.

5.3 Dependence via Occurrence Random Variables

The Model proposed by Cossette et al. (2002) is a special case of Models 5.1 and 5.2. In this model dependence via occurrence random variables I_{jk}, $j = 1, \ldots, m$ and $k = 1, \ldots, n_j$, causes dependence between risks X_{jk}, $j = 1, \ldots, m$ and $k = 1, \ldots, n_j$, represented as $X_{jk} = I_{jk} B_{jk}$. We suppose that the occurrence of a claim for the kth policy in the jth class is a function of the individual, the class, and the global risk factors. Independent random variables J_{jk}, J_j and J_0 refer to these three risk factors, respectively. Random variable I_{jk}, $j = 1, \ldots, m$, and $k = 1, \ldots, n_j$ is defined as:

$$I_{jk} = \min(J_{jk} + J_j + J_0, 1)$$

Where J_{jk}, J_j, and J_0 are independent Bernoulli random variables with

$$P(J_{jk} = 1) = \tilde{q}_{jk} \text{ and } P(J_{jk} = 0) = \tilde{p}_{jk} = 1 - \tilde{q}_{jk}$$

$P(J_{ji} = 1) = \tilde{q}_j$, $P(J_{ji} = 0) = \tilde{p}_j = 1 - \tilde{q}_j$, $P(J_0 = 1) = \tilde{q}_0$, and $P(J_{J0} = 0) = \tilde{p}_0 = 1 - \tilde{q}_0$,

The random variables I_{jk} are Bernoulli distributed based on its definition, and as a result the random vector $\underline{I} = (I_{11}, ..., I_{1n_1}, ..., I_{m1}, ..., I_{mn_m})$ would have dependent components. The probability generation function (pgf) of I_{jk} is defined as follows:

$$P_{I_{jk}}(t) = \sum_{I_{jk}=0}^{1} P(I_{jk}) t^{I_{jk}} = p_{jk} + q_{jk} t,$$

Where $p_{jk} = \tilde{p}_0 \tilde{p}_j \tilde{p}_{jk}$ and $q_{jk} = 1 - (1 - \tilde{q}_0)(1 - \tilde{q}_j)(1 - \tilde{q}_{jk})$. If $\tilde{q}_j = \tilde{q}_0 = 0$, $j = 1, ..., m$, then $q_{jk} = \tilde{q}_{jk}$ which is a special case of the individual risk model. If $\tilde{q}_j = 0$ then the portfolio will get only one class in this case. So, based on the definition of I_{jk}, the random vector $\underline{X} = (X_{11}, ..., X_{1n_1}, ..., X_{m1}, ..., X_{mn_m})$ has dependent components. The moment generating function (mgf) of each X_{jk}, $j = 1, ..., m$ and $k = 1, ..., n_j$, is:

$$M_{X_{jk}}(t) = P_{I_{jk}}\left(M_{B_{jk}}(t)\right) = p_{jk} + q_{jk} M_{B_{jk}}(t)$$

For finding moment generating function (mgf) of the aggregate claim amount S, we need to obtain the multivariate mgf of the random vector $\underline{X}$ which in turn is a function of the pgf of the random vector $\underline{I}$. The pgf of $\underline{I}$ is given by:

$$P_{\underline{I}}(\underline{t}) = \tilde{p}_0 \left[\prod_{j=1}^{m} \left(\tilde{q}_j \prod_{k=1}^{n_j} t_{jk} + \tilde{p}_j \prod_{k=1}^{n_j} P_{I_{jk}}(t_{jk}) \right) \right] + \tilde{q}_0 \prod_{j=1}^{m} \prod_{k=1}^{n_j} t_{jk}$$

Where $t = (t_{11}, ..., t_{1n_1}, ..., t_{m1}, ..., t_{mn_m})$. The multivariate mgf of $\underline{X}$ using the above equation is

$$M_{\underline{X}}(t) = P_{\underline{I}}(M_{B_{11}}(t_{11}), \ldots, M_{B_{1n_1}}(t_{1n_1}), \ldots, M_{B_{m1}}(t_{m1}), \ldots, M_{B_{mn_m}}(t_{mn_m}))$$

Given the above equation, we can use the following Lemma to find the mgf of S.

Lemma 5.3.1 [Cossette et al. (2002)]: Let $M_{Y_1,...,Y_n}(t_1, ..., t_n)$ be the multivariate mgf of the vector $(Y_1, ..., Y_n)$ given by $M_{Y_1,...,Y_n}(t_1, ..., t_n) = E[e^{t_1 Y_1} \ldots e^{t_n Y_n}]$. Then, the mgf of $Z = Y_1 + ... + Y_n$ is $M_Z(t) = M_{Y_1,...,Y_n}(t, ..., t)$.

From **Lemma 5.3.1** and the last equation, the mgf of S is:

$$M_S(t) = \tilde{p}_0 \left[\prod_{j=1}^{m} \left(\tilde{q}_j \prod_{k=1}^{n_j} M_{B_{jk}}(t) + \tilde{p}_j \prod_{k=1}^{n_j} P_{J_{jk}} M_{B_{jk}}(t) \right) \right] + \tilde{q}_0 \prod_{j=1}^{m} \prod_{k=1}^{n_j} M_{B_{jk}}(t)$$

From the above equation, we can see that F_s is a convex combination of two cumulative distribution functions, F_U and F_V:

$$F_S(x) = \tilde{p}_0 F_U(x) + \tilde{q}_0 F_V(x),\ x \geq 0 \quad (5)$$

Where U and V are random variables with the following mgf:

$$M_U(t) = \prod_{j=1}^{m} \left(\tilde{q}_j \prod_{k=1}^{n_j} M_{B_{jk}}(t) + \tilde{p}_j \prod_{k=1}^{n_j} P_{J_{jk}} M_{B_{jk}}(t) \right)$$
and $M_V(t) = \prod_{j=1}^{m} \prod_{k=1}^{n_j} M_{B_{jk}}(t)$

The cumulative distribution functions of U and V are:

$$F_U = F_{C_1} * \ldots F_{C_m}$$
$$F_V = F_{B_{11}} * \ldots * F_{B_1 n_1} * \ldots * F_{B_{m1}} * \ldots * F_{B_{mn_m}}$$

Where

$$F_{C_j} = \tilde{q}_j (F_{B_{j1}} * \ldots * F_{B_{jn_j}}) + \tilde{p}_j (F_{D_{j1}} * \ldots * F_{D_{jn_j}}),\ j = 1, ..., m,$$
$$F_{D_{jk}} = \tilde{p}_{jk} \Delta_0 + \tilde{q}_{jk} F_{B_{jk}} \quad k = 1, ..., n_j$$

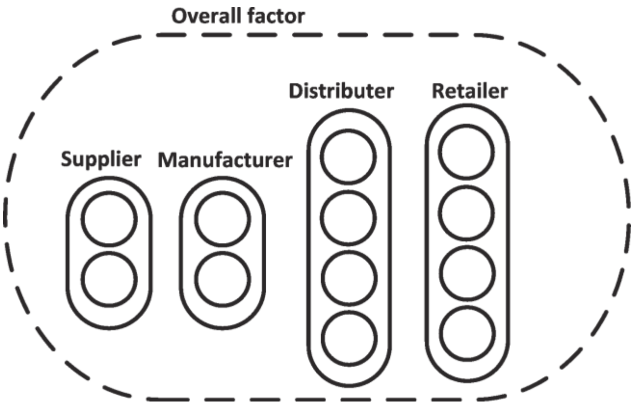

Figure 6 Schematic representation of model 5.3 or 5.2 in supply-chain. Each circle represents a company/facility

Where symbol "$*$" denotes the convolution product between two cumulative distribution functions, and Δ_d is the Dirac function:

$$\Delta_d(x) = \begin{cases} 1, & \text{if } x \geq d, \\ 0, & \text{otherwise} \end{cases}$$

In most cases like Model 5.3 we cannot find an explicit form for F_S. Therefore, we need to apply numerical approximation. In this model, given Eq. (5), two steps can be made to evaluate F_s numerically. First, with the appropriate formulas given in (6) and (7) F_U and F_V are computed. Then, F_s is calculated with (5).

In this model dependence between the occurrence random variables I_{jk}, $j = 1, ..., m$, and $k = 1, ..., n_j$, leads to dependent risks X_{jk}, $j = 1, ..., m$, and $k = 1, ..., n_j$. But, Cossette et al. (2002) assumed that all three risk factors (J_{jk}, J_j, and J_0) used in the definition of I_{jk} are independent Bernoulli random variables which can be considered as a restriction to this model. In addition, there isn't any explicit form for F_s, and we must use a numerical approximation to calculate F_s.

Application of Model 5.3 in Supply Chains

One possible example of this model is the supply chain of Toyota for car model "Prius" in which the supplier, manufacturer, retailer, and distributor are all in Japan. All four links considered as group risk factors in this model must have a close relationship to manufacture a vehicle and send it to customers. So, dependence between these business partners is required. In addition, collaboration and shared principles within companies (individual risk factors) in each group, like suppliers, is necessary to implement various activities.

5.4 Common Mixture Model

In a second model of Bauerle and Muller (1998) we compare the portfolios based on the number of external mechanisms that influence them. Assume that there are two n dimensional random vectors X and Y with the structure:

$$(X_1, \ldots, X_n) = (g_1(Z_1, W), \ldots, g_n(Z_n, W))$$
$$(Y_1, \ldots, Y_n) = (\tilde{g}_1(U_1, V, W), \ldots, \tilde{g}_n(U_n, V, W))$$

Where $Z_1, \ldots, Z_n, U_1, \ldots, U_n$ are i.i.d. random variables and (V, W) is a random vector independent of Z_k and U_k. Also, $g : R^2 \to R$ and $\tilde{g}: R^3 \to R$ are such that for every fixed W and all $k = 1, \ldots, n$, we have:

$$g_k(Z_k, W) \overset{d}{=} \tilde{g}_k(U_k, V, W)$$

i.e., they have the same distribution.

Let $S = \sum_{k=1}^{n} X_k$ and $S' = \sum_{k=1}^{n} Y_k$. In the following theorem Bauerle and Muller (1998) show that given some conditions on functions $\tilde{g}_i$, the portfolio Y is riskier that the portfolio X

Theorem 5.4.1. [Bauerle and Muller (1998)] If the functions $\tilde{g}_i$ is increasing in the second argument, then:

a) $X \leq_{symsm} Y$
b) $S \leq_{sl} S'$.

There is more dependence in portfolio Y than in X due to the extra environmental variable V. So, the external mechanism V, which has a common influence on all risks in portfolio Y is an important risk factor.

As a special case if we assume that W is constant then $Y_k = \tilde{g}_k(U_k, V)$ and $X_k = g_k(Z_k)$. Then, Bauerle and Muller (1998) obtain the following corollary of **Theorem 5.4.1.**

Corollary 5.4.2. [Bauerle and Muller (1998)] Let V be any random variable and let $Y = Y_1, \ldots, Y_n$ be a random vector such that $Y_1, \ldots, Y_n$ are conditionally independent given $V = v$ and such that the conditional distributions $P(Y_k \mid V = v)$ are stochastically increasing in v for all $k = 1, \ldots, n$. Moreover, let $X = (X_1, \ldots, X_n)$ be a random vector of independent random variables with the same marginal distribution as Y. Then

$$X \leq_{sm} Y \text{ and } S = \sum_{k=1}^{n} X_k \leq_{sm} S' = \sum_{k=1}^{n} Y_k$$

In the following theorem Shaked and Shanthikumar (1997) consider the same assumptions as **Corollary 5.4.2.**

Theorem 5.4.3. [Shaked and Shanthikumar (1997)] Let $X = (X_1, X_2, \ldots, X_n)$ be a conditionally increasing in sequence random vector and let $Y = (Y_1, Y_2, \ldots, Y_n)$ be a random vector of independent random variables such that $X_k =_{st} Y_k$, $k = 1, \ldots, n$. Then $Y \leq_{sm} X$

In contrast to **Model 5.1**, in this model, the comparison of two portfolios is based on the number of external mechanisms which affect the portfolios. In this model we assume that all individual random variables $Z_1, \ldots, Z_n$ and $U_1, \ldots, U_n$ are independent. Also, (V, W) is a random vector independent of Z_i and U_i. But the productivity of a network is based on the contribution of individual factors in each group and collaboration of business partners. Therefore, the assumption of independence can be considered as a restriction for this model.

Application of Model 5.4 in Supply Chains:

In **Figure 7** Network 2 is affected by external mechanisms. So, the dependence between risks in Network 2 is more than that in Network 1. As a result, we can conclude that Network 2 is riskier than Network 1. If we assume the supply chain of Toyota as Network 2 and the supply chain of Honda as Network 1. The severe natural disaster that occurred in northeastern Japan on March 11, 2011, hit both supply chains. But the powerful tsunami triggered by the earthquake had a very bad effect on some plants of Toyota in Miyako. So, we can say that the supply chain of Toyota is riskier than that of Honda because of the geographic location of its plants.

5.5 Two-Point Distribution Model

5.5.1 Indistinguishable Individuals (Bauerle and Muller (1998))

When permutation does not have any effect on the joint distribution of the random vector of n risks of a portfolio, we can say that these individual risks are indistinguishable individuals. This implies that the marginal distribution is the same for all risks, i.e., there is a $p \in (0, 1)$ and some $\alpha > 0$ such that $P(X_k = 0) = p = 1 - P(X_k = \alpha)$ for all $k = 1, \ldots, n$. In probability theory a sequence of such random variables is said to be exchangeable (or interchangeable). Suppose that $\alpha = 1$ then random variables $X_1, X_2, \ldots, X_n$ establish a sequence of exchangeable Bernoulli variables.

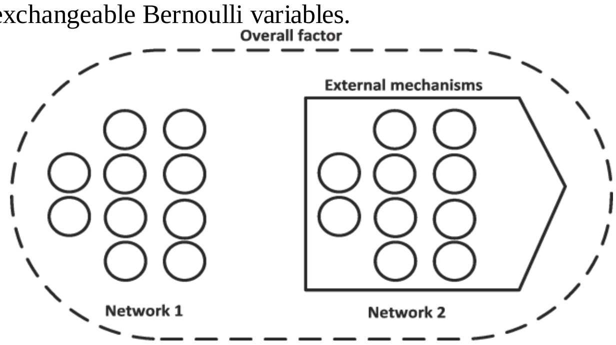

Figure 7 Schematic representation of model 5.4 in supply-chain. Each circle represents a company/facility

Hence if we assume S_n to be the total amount of claims in a portfolio of n risks, which occur from a sequence of exchangeable Bernoulli variables then based on De Finetti theorem, S_n is a mixture of binomial distributions, i.e.,

$$P(S_n = K) = \int_0^1 \binom{n}{k} \gamma^k (1 - \gamma)^{n-k} F(d\gamma)$$

Where γ is a dependence parameter, which continuously varies between independence and maximal dependence and F is the probability distribution of γ over [0, 1]. In fact, F is a prior for random vector γ given $\Gamma = \gamma$. Bauerle and Muller (1998) in the following theorem show how the riskiness of the portfolio (S_n) is influenced by mixing distribution F.

Theorem 5.5.1. [Bauerle and Muller (1998)] Let $S_n(S'_n)$ be the total claim amount of a portfolio of n risks, which stem from a sequence of exchangeable Bernoulli variables with mixing distribution $F(F')$. Then $F \leq_{sl} F'$ implies $S_n \leq_{sl} S'$.

In contrast to Models 5.1 and 5.4 of Bauerle and Muller (1998), all risks should have the same marginal distribution here. The total claim S_n in this case is a mixture of binomial distributions. But we did not talk about the distribution of S in the former models.

Application of Model 5.5.1 in Supply Chains

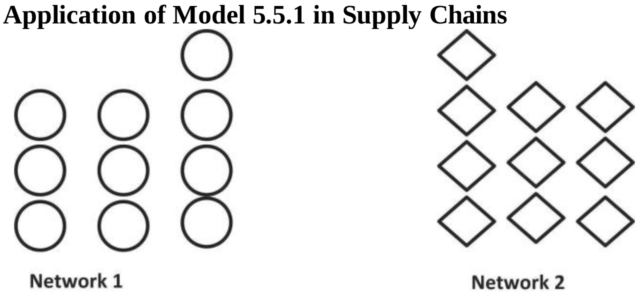

Figure 8 Schematic representation of model 5.5.1 in a supply chain. Each circle represents a company, and each diamond indicates a facility. If we assume that the prior probability of dependence factor in Network 2 is greater than Network 1, then Network 2 becomes riskier.

Consider **Figure 8** consisting of Network 1 and Network 2. In this model we assume that individual risks (shown with circles in Network 1 and with diamonds in Network 2) are a sequence of exchangeable Bernoulli variables. In Network 1 (Network 2) each risk factor is 1 if it produces any loss with probability of p (or $\acute{p}$ for Network 2) otherwise is 0 with probability of $1 - p$ (or $1 - \acute{p}$ for Network 2). But the quantities p and $\acute{p}$ are unknown. If $F(F')$ expresses the probability distribution about uncertainty of p(p′) then based on the result of **Theorem 5.5.1** we can conclude that Network 2 is riskier than Network 1 if the prior of p′ (i.e., F') is greater than the prior of p (i.e., F) in stop-loss order.

5.5.2 Distinguishable Individuals

The additive damage model proposed by Bauerle and Muller (1998) is a well-known model in probability theory. In this model we are dealing with two sources which produce some normally distributed damage. One source has the same effect on all individuals, while the impact of the other one depends on the individual behaviour of everyone. We will have the claim amount of β, if the sum of these two damages surpasses some level z_k.

The distribution function of the distinguishable individual's model which is constructed based on model 5.4 assumes only two values. Let $N(\mu, \sigma^2)$ denote a univariate normal distribution with mean μ and variance $\sigma^2 > 0$. $N(\mu, 0)$ indicates a one-point distribution in μ. If X ~ N (0, 1), then assuming $P(X \leq z_p) = p$, z_p would be the p-quantile of the standard normal distribution. Now we consider model 5.4 with $W \sim N(0, \sigma^2), V \sim N(0, \tau^2 - \sigma^2), Z_k \sim N(0, 1 - \sigma^2)$ and $U_k \sim N(0, 1 - \tau^2)$ when $0 \leq \sigma^2 < \tau^2 \leq 1$. All random variables should be independent in this model. We define:

$$g_k(z, w) = \begin{cases} \beta_k, & z + w \geq z_{p_k} \\ 0, & \text{else} \end{cases}$$

and

$$\tilde{g}_k(u, v, w) = \begin{cases} \beta_k, & u + v + w \geq z_{p_k} \\ 0, & \text{else} \end{cases}$$

Recall from model 5.4 that $X_k = g_k(Z_k, W)$ and $Y_k = \tilde{g}_k = (U_k, V, W)$ for $k = 1, \ldots, n$. It would be easy to show that all conditions in Model 5.4 are held for this model, so we can say that $X \leq_{sl} Y$ and therefore X is less risky than Y. This model is an extension of model 5.4. But distributions and functions of all risks assume only two values. Also, all random variables Z, U, W, V are normally distributed, and the effect of additive damage depends on the behaviour of everyone.

Application of Model 5.5.2 in Supply Chains

Assume that in Network 1 each risk is a function of an individual risk factor (company) and an overall risk factor (natural disaster). See Fig. 9 . The first source of damage (natural disaster) influences all companies in the same manner. But the occurrence of the second source of risk, like inventory decline, depends on the individual behaviour of each company, such as how fast it recovers from a disaster. So, the loss amount of β_k in company k occurs if the sum of these damages (natural disaster and inventory decline) gets larger than a specific amount. In Network 2 each risk is a function of an individual risk factor (company), an external mechanism (demand change) and an overall risk factor (natural disaster). The effect of the natural disaster and the demand change are the same for all companies. But each company faces an inventory decline, if it cannot manage the crisis properly. If total damage arising from these sources exceeds some level z_{p_k}, then the company tolerates a loss of β_k. Since the construction of this model is based on Model 5.4, we can apply the result of **Theorem 5.4.1** to conclude that Network 2 is riskier than Network 1 due to the additive damage.

6. DISTRIBUTION FUNCTION (COPULA) BASED DEPENDENCY MODELS

In this section we consider two copula-based models: one for a single group and the other for multiple groups.

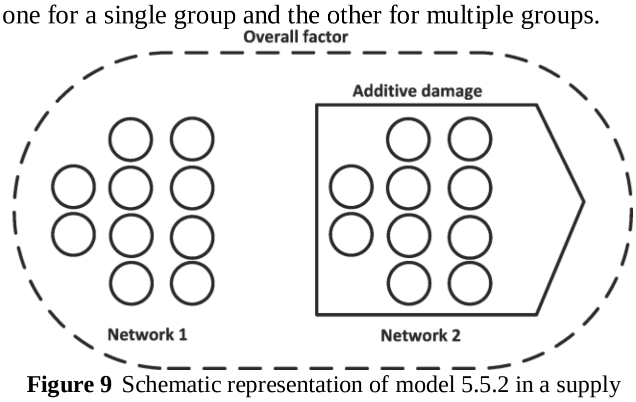

Figure 9 Schematic representation of model 5.5.2 in a supply chain. Each circle is a representation of a Bernoulli random variable that is associated with a company/facility

6.1 Copula based Dependent Single-group Risks Model (Genest et al. (2003))

In the single-class portfolio problem, where each individual risk may be represented in the form $X_k = J_k B_k$ with indicator Bernoulli random variable J_k with expectation q_k, using copula is another way of modelling dependence among the components of the vector $J = (J_1, ..., J_n)$. We can write the joint cumulative distribution function of J as follows:

$$F_J(j_1, ..., j_n) = C\{F_{J_1}(j_1), \dots, F_{J_n}(j_n)\}, \quad j_k = 0 \text{ or } 1, \; k=1, ..., n \tag{8}$$

Where, $F_{J_k}(t) = P(J_k \le t)$ is the marginal cumulative distribution function of J_k with $F_{J_k} = 0$, for $j_k \le 0$. C: $[0, 1]^n \rightarrow [0, 1]$ is a copula. All marginals F_{J_k} 's of a cumulative distribution function are uniformly distributed on the unit interval. We can say that copula is the distribution function of a random vector with uniform marginals. The advantage of writing F_J as in (8) is that it allows us to separate the definition of the marginals $F_{J_1}, F_{J_2}, ..., F_{J_n}$ and the definition of the dependence which is constructed through the copula $C(u_1, ..., u_n)$. Various copulas can be found in the literature. The simplest one is the independent copula:

$$C^{Ind}(u_1, ..., u_n) = u_1 \times \ldots \times u_n.$$

See Genest et al. (2003) and Cossette et al. (2002) for some examples of copulas.

6.2 Copula based Dependent Multi-group Risks Model (Genest et al. (2003))

This section is a multi-class extension of the Archimedean model of the previous section. Assume that the joint distribution function of $(n_1 + ... + n_m)$-dimensional vector $J = (J'_1, \dots, J'_m)$ of occurrence random variables is written as:

$$F_J(j'_1, \dots, j'_m) = \int_0^\infty \cdots \int_0^\infty \prod_{j=1}^{m} \prod_{k=1}^{n_j} \left\{F^*_{J_{jk}}\left(j'_{jk}\right)\right\}^{\theta_j} dM(\theta_1, \dots, \theta_m),$$

Where M is the m-variate distribution function of the latent vector $\Theta = (\theta_1, ..., \theta_m) \in [0, \infty)$ and $F^*_{J_{jk}} = \exp\left[-\phi_j\left\{F_{J_{jk}}\right\}\right]$ is the cumulative distribution function of Bernoulli random variable J_{jk} with expectation $1 - \exp\{-\phi_j(1 - q_{jk})\}$, $1 \le k \le n_j$ and $1 \le j \le m$. The J_{jk}'s are mutually independent Bernoulli random variables, conditioned on the value of Θ with

$$E(J_{jk}|\Theta = \theta) = E(J_{jk}|\Theta_j = \theta_j) = 1 - \exp\{-\theta_j \phi_j(1 - q_{jk})\} = q_{jk\theta_j}.$$

We assume as before that the B_{jk}'s are independent among themselves and from all indicators. Let $J^{(\theta)}_{jk}$ be mutually independent indicator random variables with mean $q_{jk\theta_j}$ for all $1 \le k \le n_j$ and $1 \le j \le m$. Considering Eq. (9), the sums $S^{(\theta)}_j$ and $S^{(\theta)}$ is defined as in Eq. (4).

$$X^{(\theta)}_{jk} = X^{(\theta_j)}_{jk} = J^{(\theta_j)}_{jk} B_{jk} \text{ (Given } \Theta = \theta)$$

As S^{θ} is composed of mutually independent terms, the total claim distribution F_S may again be calculated in the form, Eq. (3).

In this model as in model 5.3 we assume that occurrence random variables J_{jk}, $j = 1, ..., m$ and $k = 1, ..., n_j$, are dependent. But, we use copula to separate the definition of the dependence between components of $\underline{J} = (J_{11}, ..., J_{1n_1}, ..., J_{mn_1}, ..., J_{mn_m})$ and definition of the marginals $F_{J_{jk}}$.

Even though the models reviewed in Chapter 2 provide valuable insight, some assumptions they require do not necessarily fit with real life supply chains. One of the strongest assumptions may be the independence between the individual risks. Therefore, in the next section we extended the existing models and removed the assumption of individual risks independence.

7. EXTENSION: DEPENDENT INDIVIDUAL RISK FACTORS

Lemma 7.0.1. [Tchen (1980)]: Suppose there are two n-dimensional random vectors Z and U where the distribution function of U is Frechet upper bound. Then $Z \le_{sm} U$. (Proof is in **Theorem 5.(a)** of Tchen (1980))

Theorem 7.0.2. Consider n-dimensional random vectors Z and Frechet upper bound distributed U such that $Z \le_{sm} U$ (**Lemma 7.0.1**) and let W be a m-dimensional vector, which is independent from Z and U. Then,

$$Z \le_{sm} U \Rightarrow (g_1(Z_1, W), g_2(Z_2, W), ..., g_n(Z_n, W)) \le_{sm} (g_1(U_1, W), g_2(U_2, W), ..., g_n(U_n, W))$$

Where $g_k(z, w)$'s, $k = 1, 2, ..., n$, are all monotone in z for every w. If we suppose the following constructions for X and Y

$$X = (g_1(Z_1, W), g_2(Z_2, W), ..., g_n(Z_n, W))$$
$$Y = (g_1(U_1, W), g_2(U_2, W), ..., g_n(U_n, W))$$

Then $X \le_{sm} Y$ results in $S \le_{sl} S'$ (**Theorem 2.6** of Bauerle and Muller (1998)).

Proof: Since $Z \le_{sm} U$ and $g_k(z, w), k = 1, \ldots, n$ are monotone functions in z for every w, from **Theorem 2.2(a)** of Shaked and Shanthikumar (1997) it follows that

$$(Z_1, ..., Z_n) \le_{sm} (U_1, ..., U_n)$$
$$\Rightarrow (g_1(Z_1, W|W = w), ..., g_n(Z_n, W|W = w)) \le_{sm} (g_1(U_1, W|W = w), ..., g_n(U_n, W|W = w))$$

The super modular stochastic order is closed under mixtures (**Theorem 2.2(d)** of Shaked and Shanthikumar (1997)). Hence

$$(g_1(Z_1, W), ..., g_n(Z_n, W)) \le (g_1(U_1, W), \ldots, g_n(U_n, W)) \Rightarrow X \le_{sm} Y.$$

Application of model 7 in supply chains

The most important aspect of **Figure 10** compared to other figures is the connection between companies (individual risks factors). It means that in this model we relax the assumption of independence between individual risks. Since cycles in Networks (1 and 2) of **Figure 10** are not important we can call this structure a tree structure. Considering all assumptions of the last model, we can conclude that Network 1 is less risky than Network 2 due to the existence of less risky companies in Network 1 compared to Network 2.

8. CONCLUSION

In this chapter we have reviewed and extended the literature on risk dependency modelling. All models of dependencies reviewed in this chapter are summarized in **Table 3.**

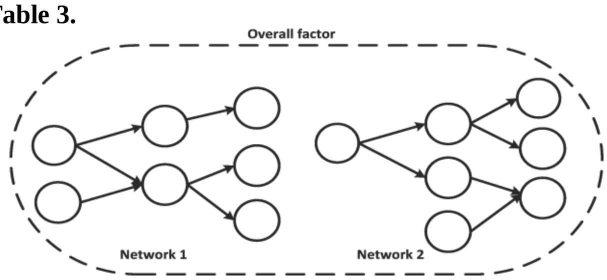

Figure 10 Schematic Representation of model 7 in a supply chain. Each circle represents a company/facility, and connections denotes dependence

Table 3 Comparison of reviewed risk dependency models

Type of Dependence	Reference
Multi-groups Dependence between groups	5.1: Bauerle and Muller (1998), 5.3:Cossette *et al*. (2002), 5.2: Genest et al. (2003), 6.2: Genest *et al*. (2003), 5.4: Bauerle and Muller (1998), 5.5.2: Bauerle and Muller (1998)
Multi-groups Dependence within groups	Removing overall risk factor V in 5.1: Bauerle and Muller (1998), Let $\tilde{q} = 0$ in 5.3: Cossette *et al*. (2002), Let J = 0 in 5.2: Albers (1999)
Single-group Dependent risks	Let $\tilde{q}_j = 0$ in 5.3: Cossette *et al*. (2002), 4.1: Goovaerts and Dhaene (1996), 4.2: Dhaene and Goovaerts (1997), 6.1: Genest *et al*. (2003)
Dependent individual risk factors	7: Beigi and Hassini (2013)

We have also shown how these models can be applied in a supply chain environment. We have indicated the shortcomings of these models in relation to supply chain environments. These present possible future research venues in this area. Specifically, opportunities exist to study the modelling of risk dependencies where group specific risk factors in a multi-class dependency model are dependent. Another possible research venue is to develop models for risk prediction and propagation in supply chain networks in the presence of dependencies. Finally, with the availability of big data in supply chains it will be interesting to empirically test the validity of the dependency models of risks in supply chains.

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APPENDIX 1: PROOF OF THE THEOREM 4.2.1

The possible outcome for S will be: $0, \alpha_1, \alpha_1+\alpha_2, \alpha_1+\alpha_2+\alpha_3, \dots, \alpha_1+\dots+\alpha_n$. Therefore, we will have:

$$\begin{aligned}
\Pr(S=0) &= \Pr(X_1=0; X_2=0,\dots,X_n=0) = \Pr(X_1=0) = \qquad (A.9)\\
p_1, \Pr(S=\alpha_1+\alpha_2+\dots+\alpha_k) &= \Pr(X_1=\alpha_1; X_2=\alpha_2;\dots;X_k=\alpha_k;X_{k+1}=0;\dots;X_n=0)\\
&= \Pr(X_k=\alpha_k; X_{k+1}=0)\\
&= \Pr(X_k=\alpha_k).\Pr(X_{k+1}=0|X_k=\alpha_k)\\
&= p_{k+1}-p_k, \qquad k=1,2,\dots,n-1,\\
\Pr(S=\alpha_1+\dots+\alpha_n) &= \Pr(X_1=\alpha_1,\dots,X_n=\alpha_n) = \Pr(X_n=\alpha_n) = 1-p_n.
\end{aligned}$$

Then, the distribution function of *S* will be as follows:

$$F_S(s) = \begin{cases} p_1, & 0 \le s \le \alpha_1 \\ p_{k+1}, & \alpha_1+\cdots+\alpha_k \le s \le \alpha_1+\cdots+\alpha_{k+1}, \quad k=1,2,\dots,n-1. \\ 1, & s \ge \alpha_1+\cdots+\alpha_n \end{cases}$$

APPENDIX 2: PROOF OF THE THEOREM 5.2.1

Define $E(J_0)=r$, $E\left(J_j^{(\alpha)}\right)= r_j^{(\alpha)}$ and $E\left(J_{jk}^{(\alpha\beta)}\right)= r_{jk}^{(\alpha\beta)}$ and similarly $\bar{r}, \bar{r}_j^{(\alpha)}$ and $\bar{r}_{jk}^{(\alpha\beta)}$. Using conditions (a) and (b), for every choice of $1 \le k \le n_j$ and $1 \le j \le m$ we have:

$$\begin{aligned}
F_{X_{jk}}(x) &= r\left\{r_j^{(1)}F_{X_{jk}^{(11)}}(x) + \bar{r}_j^{(1)}F_{X_{jk}^{(10)}}(x)\right\} + \bar{r}\left\{r_j^{(0)}F_{X_{jk}^{(01)}}(x) + \bar{r}_j^{(0)}F_{X_{jk}^{(00)}}(x)\right\}\\
&= r\left[r_j^{(1)}\left\{\bar{r}_{jk}^{(11)} + r_{jk}^{(11)}F_{B_{jk}^{(11)}}(x)\right\} + \bar{r}_j^{(1)}\left\{\bar{r}_{jk}^{(10)} + r_{jk}^{(10)}F_{B_{jk}^{(10)}}(x)\right\}\right]\\
&\quad + \bar{r}\left[r_j^{(0)}\left\{\bar{r}_{jk}^{(01)} + r_{jk}^{(01)}F_{B_{jk}^{(01)}}(x)\right\} + \bar{r}_j^{(0)}\left\{\bar{r}_{jk}^{(00)} + r_{jk}^{(00)}F_{B_{jk}^{(00)}}(x)\right\}\right], x \ge 0
\end{aligned}$$

After a rearrangement of the terms, we get:

$$F_{X_{jk}}(x) = \bar{q}_{jk} + rr_j^{(1)}r_{jk}^{(11)}F_{B_{jk}^{(11)}}(x) + r\bar{r}_j^{(1)}r_{jk}^{(10)}F_{B_{jk}^{(10)}}(x) + \bar{r}r_j^{(0)}r_{jk}^{(01)}F_{B_{jk}^{(01)}}(x) + \bar{r}\bar{r}_j^{(0)}r_{jk}^{(00)}F_{B_{jk}^{(00)}}(x), \quad x \ge 0 \qquad (A.10)$$

Where

$$q_{jk} = rr_j^{(1)}r_{jk}^{(11)} + r\bar{r}_j^{(1)}r_{jk}^{(10)} + \bar{r}r_j^{(0)}r_{jk}^{(01)} + \bar{r}\bar{r}_j^{(0)}r_{jk}^{(00)} = P(X_{jk} \neq 0). \qquad (A.11)$$

Accordingly, each risk $X_{jk} = J_{jk}B_{jk}$ may be denoted as it follows:

$$X_{jk} = \begin{cases} B_{jk}, & J_{jk}=1 \\ 0, & J_{jk}=0 \end{cases}$$

Based on the definition of indicator random variable J_{jk} with $P(J_{jk}=1) = q_{jk}$ in Eq. (A.11),

$$\begin{aligned}
F_{X_{jk}}(x) &= \Pr(X_{jk} \le x) = \Pr(B_{jk}J_{jk} \le x) \qquad (A.12)\\
&= \Pr\{B_{jk}J_{jk} \le x | J_{jk}=0\}\Pr(J_{jk}=0)\\
&\quad + \Pr\{B_{jk}J_{jk} \le x | J_{jk}=1\}\Pr(J_{jk}=1) = (1-q_{jk})\Pr\{x \ge 0\} + q_{jk}\Pr\{B_{jk} \le x\}\\
&= \bar{q}_{jk} + q_{jk}F_{B_{jk}}(x)
\end{aligned}$$

From (A.10) and (A.12), the distribution of the strictly positive random variable B_{jk} is a mixture distribution

$$F_{B_{jk}} = \frac{rr_j^{(1)} r_{jk}^{(11)}}{q_{jk}} F_{B_{jk}^{(11)}} + \frac{r\bar{r}_j^{(1)} r_{jk}^{(10)}}{q_{jk}} F_{B_{jk}^{(10)}} + \frac{\bar{r}r_j^{(0)} r_{jk}^{(01)}}{q_{jk}} F_{B_{jk}^{(01)}} + \frac{\bar{r}\bar{r}_j^{(0)} r_{jk}^{(00)}}{q_{jk}} F_{B_{jk}^{(00)}}$$

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